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THE LIAPUNOV FUNCTION FOR THE SECOND ORDER DIFFERENCE SYSTEMS*

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The Liapunov function, whose fixed sign property together with its first difference is explicitly defined by the conditions of asymptotic stability and instability, is derived for a stationary difference system of second order. Examples are given of the use of that function in the analysis of nonlinear and nonstationary problems.

1. Consider the system of difference equations

$$x_{n+1} = ax_n + by_n, \quad y_{n+1} = cx_n + dy_n \tag{1.1}$$

where $(a, b, c, d$ are constant coefficients. For this system we construct the following Liapunov function:

$$V(x, y) = [by - (d - k)x]^2 + [cx - (a - k)y]^2 + (1 - k^2)(x^2 + y^2) \; (k = a + d / \Delta + 1, \;\; \Delta = ad - cb) \tag{1.2}$$

The first difference ΔV by virtue of system (1.1) is of the form

$$(\Delta V)_{(1.1)} = V(ax + by,\; cx + dy) - V(x, y) = -(1 - \Delta^2)(1 - k^2)(x^2 + y^2), \quad \Delta + 1 \neq 0 \tag{1.3}$$

When the inequalities

$$1 - \Delta^2 > 0, \quad 1 - k^2 > 0 \tag{1.4}$$

are satisfied, function $V(x,y)$ is positive definite, and $\Delta V(x,y)$ is negative definite.

On the basis of the analog of the theorem on asymptotic stability the motion $x = y = 0$ in the case of difference systems is asymptotically stable /1–3/. Hence conditions (1.4) are sufficient conditions of asymptotic stability of the trivial solution of system (1.1). Since the fulfillment of one of inequalities (1.4) or both of them with opposite signs makes V of varying sign, or ΔV becomes of fixed sign, the same as that of V. Hence the solution $x = y = 0$ of system (1.1) is unstable.

Note that functions V and ΔV have a meaning, when conditions $\Delta + 1 \neq 0$ are satisfied. If $\Delta + 1 = 0$ and $a + d \neq 0$, the solution $x = y = 0$ of system (1.1) is unstable, since then a function of the form

$$V = dx^2 - (c + b)xy + ay^2$$

which is of varying sign, can be taken as the Liapunov function. Its first difference by virtue of system (1.1) is

$$(\Delta V)_{(1.1)} = -(a + d)(x^2 + y^2)$$

which indicates that the trivial solution is unstable. When inequalities (1.4) are satisfied with the equality sign, we have either instability or a weak (nonasymptotic) stability. Thus, when $\Delta + 1 \neq 0$, (1.4) represents the necessary and sufficient conditions of asymptotic stability of solution $x = y = 0$ of system (1.1).

Function

$$V_1(x, y) = [by - (d - k)x]^2 + (1 - k^2)x^2$$

may be taken as the Liapunov function for (1.1); it is positive definite under condition (1.4) and $b \neq 0$. Under condition (1.4) the first difference ΔV_1 is of fixed sign and negative. When $b \neq 0$, the set $x = 0$ does not contain of integral half-trajectories of system (1.1), hence in conformity with the Barbashin–Krasovskii analog theorem on difference systems, the motion $x = y = 0$ of system (1.1) is asymptotically stable.

2. Let us estimate V and ΔV, using the explicit expression of V in terms of coefficients of the system and the analytic expressions in conditions (1.4). We introduce the notation

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$$m = 1 - k^2, \quad M = b^2 + (a-k)^2 + c^2 + (d-k)^2$$
$$M_1^2 = [b^2 + (d-k)^2 - c^2 - (a-k)^2]^2 + 4\,[b\,(d-k) + c\,(a-k)]^2,$$
$$M_1 < M$$
$$g = m + \frac{M - M_1}{2}, \quad h = m + \frac{M + M_1}{2}$$

and set $x = x_1,\ y = x_2,\ x = (x_1, x_2),\ \|x\| = (x_1^2 + x_2^2)^{1/2}$.

We have the estimates

$$g\,\|x\|^2 \leqslant V(x, y) \leqslant h\,\|x\|^2, \quad (\Delta V)_{(1.1)} = -(1 - \Delta^2)m\,\|x\|^2$$

If x' and x'' are two arbitrary points, then

$$|V(x'') - V(x')| \leqslant (M + m)\,\|x'' - x'\|\,(\|x'' - x'\| + 2\,\|x'\|)$$

3. Using function (1.2) for V and the estimates for V and ΔV, then, applying conventional methods, we obtain

$$x_n^2 + y_n^2 \leqslant hg^{-1}(1 - q_1)^n \cdot (x_0^2 + y_0^2), \quad q_1 = (1 - \Delta^2)mh \tag{3.1}$$
$$x_n^2 + y_n^2 \geqslant gh^{-1}(1 - q_2)^n (x_0^2 + y_0^2), \quad q_2 = (1 - \Delta^2)mg$$

From (3.1) we obtain the maximum time of transition of point (x_0, y_0) of a circle or arbitrary radius R toward the interior of a circle of radius $\varepsilon\ (x_n^2 + y_n^2 \leqslant \varepsilon^2)$. This time is defined by the formula

$$n \geqslant \varepsilon^2 R^{-2} g h^{-1} / \ln(1 - q_1)$$

4. Function V can be used for obtaining estimates of the attraction region of nonlinear difference systems for which (1.1) is the system of first approximation.

Let us consider the system

$$x_{n+1} = ax_n + by_n + \gamma x_n^3, \quad y_{n+1} = cx_n + dy_n + \beta y_n^3$$

where γ, β are constants, and $(|\gamma| > |\beta|)$. The attraction region is defined by the inequality

$$\|x\| \leqslant (g/h)^{1/2} r$$

where

$$r = |\gamma|^{-1/2}\left\{\left[l + \frac{(1 - \Delta^2)\,m\,|\gamma|}{M}\right]^{1/2} - l^{1/2}\right\}, \quad l = a^2 + b^2 + c^2 + d^2$$

5. Function (1.2) can also be used for analyzing the stability of difference systems of second order with variable coefficients.

Consider the second order difference equation

$$x(n + 2) + 2\,Ax(n + 1) + Bx(n) = 0 \tag{5.1}$$

where A and B are constant coefficients. This equation is equivalent to a system of form (1.1)

$$x_{n+1} = y_n, \quad y_{n+1} = -2Ay_n - Bx_n \tag{5.2}$$

for which we have the Liapunov function

$$V(x, y) = V_1(x, y) + V_2(x, y), \quad V_1(x, y) = [y + 2Ax - Cx]^2 + (1 - C^2)\,x^2, \quad V_2(x, y) = [Bx + Cy]^2 + (1 - C^2)\,y^2, \quad C = 2A/(B + 1) \tag{5.3}$$

The condition of asymptotic stability of solution $x = y = 0$ of system (5.2) are the inequalities

$$B^2 < 1, \quad (B + 1)^2 > 4A^2, \quad B + 1 \neq 0 \tag{5.4}$$

As an example of application of function V, we shall consider the following system (analysed in /4/):

$$x_{n+1} = y_n, \quad y_{n+1} = p_n y_n + a^2 x_n \tag{5.5}$$

Using function /4/ defined by

$$V = (1 + a^2)^{-1}(a^2 V_1 + V_2) = a^2 x^2 + y^2 \quad (A = 0,\ B = -a^2)$$

and the conditions (5.4) of asymptotic stability, we obtain the condition of asymptotic stability of solution $x = y = 0$ of system (5.1)

1) $1 - a^2 > 0$
2) $-[(1 - a^2 - \delta)(1 - a^2)]^{1/2} < p_n < 1 - a^2 - \varepsilon$

where δ, ε are positive constants. The obtained asymptotic stability region is wider than that obtained earlier in /4/.

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